Investigation of Generating Cubic Polynomials

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MATH 67098

December 5, 2021

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Gupta and Szymanski discuss methods to generate "nice" cubic polynomials in their article *Cubic Polynomials with Rational Roots and Critical Points* (2010). The desired polynomials will be cubic and have rational values for the roots and critical points of the function. This type of polynomial is convenient to have when teaching elementary calculus problems or even high school level algebra and precalculus so that a calculator will not be necessary to find critical values when graphing and discussing the key features of the function and graph. Since we are considering roots, local extrema, and points of inflection, we can apply the first and second derivative tests to determine the values. However, the purpose of this investigation is to reverse engineer these ideas to construct polynomials so that the zeros, extrema, and inflection point are all rational. Gupta and Szymanski did achieve a function generator which I have investigated and will expand upon as follows.

Discussion

Let us consider a cubic polynomial, $f(x) = (x-a)(x-b)(x-d)$ where $a, b,\ d \in \mathbb{Q}$ and where the roots of $f'(x), \& f''(x)$ are also members of the Rational Numbers. Let us make a translation of f to g so that two of the roots are symmetrical to the origin. Therefore, let the translation be $g(x) = f\left(x + \frac{a+b}{2}\right) = \left(x - \frac{a-b}{2}\right)\left(x - \frac{b-a}{2}\right)\left(x - d + \frac{a+b}{2}\right)$ where $g(x)$ has roots $\pm\frac{b-a}{2}, d - \frac{a+b}{2}$. Let us denote $t = \frac{b-a}{2}$ and $s = d - \frac{a+b}{2}$ so that the roots of $g(x)$ are $\pm t, s$. Then, $g(x) = (x+t)(x-t)(x-s) = (x^2 - t^2)(x-s)$. We then derive, $g'(x) = 2x(x-s) + x^2 - t^2 = 3x^2 - 2sx - t^2$ which is a quadratic function with the roots $x = \frac{2s \pm \sqrt{4s^2+12t^2}}{6} = \frac{2s \pm \sqrt{4(s^2+3t^2)}}{6} = \frac{s \pm \sqrt{s^2+3t^2}}{3}$. Additionally, we see $g''(x) = 6x - 2s$ which is linear, and the root of $g''(x)$ is $x = \frac{s}{3}$ which is rational, so we do not need to consider $g''(x)$ further. However, the roots of $g'(x)$ are rational if and only if the discriminant $D = 4s^2 + 12t^2$ is a square of rational number, that is, $\frac{D}{4} = s^2 + 3t^2 = r^2$ where $s, t, r \in \mathbb{Q}$.

Let us show next that any rational solution (s, t, r) of the equation $s^2 + 3t^2 = r^2$ is a rational multiple of a primitive integer solution of this equation. Let $s = \frac{a}{b}, t = \frac{c}{d}, r = \frac{e}{f}$ where $a, b, c, d, e, f \in \mathbb{Z}$ and $b, d, f \neq 0$. Then we have $\left(\frac{a}{b}\right)^2 + 3\left(\frac{c}{d}\right)^2 = \left(\frac{e}{f}\right)^2$. Clearing the denominators, we get $a^2d^2f^2 + 3b^2c^2f^2 = b^2d^2e^2 \rightarrow (adf)^2 + 3(bcf)^2 = (bde)^2$. We conclude that $(U, V, W) = (adf, bcf, bde)$ is an integer solution of $s^2 + 3t^2 = r^2$. Let $\gcd(U, V, W) = p$, then $(u, v, w) = \left(\frac{U}{p}, \frac{V}{p}, \frac{W}{p}\right)$ is a primitive solution of $s^2 + 3t^2 = r^2$. Now we can express our original rational solution as $(s, t, r) = \alpha(u, v, w)$, where $\alpha = \frac{p}{bdf} \in \mathbb{Q}$. We have checked that any rational solution of our equation is a rational multiple of a primitive solution, and hence we can now concentrate on describing primitive solutions.

Description of all Primitive Solutions to $u^2 + 3v^3 = w^2$

We now find primitive solutions for $u^2 + 3v^2 = w^2$, that is, $\gcd(u, v, w) = 1$. Note that we can also assume that u, v, and w are all non-negative. There are four considerations seen in the table below.

u	v	w^2	w	Outcome
even	even	even	even	Not primitive
odd	even	odd	odd	Case 1
even	odd	odd	odd	Not possible
odd	odd	even	even	Case 2

It is obvious that u, v, and w^2 cannot all be even as the solution will not be primitive. Let us explain why we cannot have even u and odd v. Let $u = 2k, v = 2l + 1, w = 2p + 1$ for some integers *k, l,* and *p*. By substitution we see:

$$u^2 + 3v^2 = w^2$$

$$(2k)^2 + 3(2l + 1)^2 = (2p + 1)^2$$

$$4k^2 + 3(4l^2 + 4l + 1) = 4p^2 + 4p + 1$$

$$4(k^2 + 3l^2 + 3l) + 3 \quad = \quad 4(p^2 + p) + 1$$

When $4(k^2 + 3l^2 + 3l) + 3$ is divided by 4 the remainder is 3. When $4(p^2 + p) + 1$ is divided by 4 the remainder is 1, thus $4(k^2 + 3l^2 + 3l) + 3 \neq 4(p^2 + p) + 1$. Hence, u = even, v = odd, w^2 = odd is not possible.

Proposition 1: Let u and w be odd relatively prime integers. Then gcd $\left(\frac{u+w}{2}, \frac{u-w}{2}\right) = 1$.

Proof. Presume $p | \frac{u+w}{2}$ and $p | \frac{u-w}{2}$ where $p \in \mathbb{Z}$ and prime for some $r, t \in \mathbb{Z}, r = \frac{u+w}{2p}, t = \frac{u-w}{2p}$.

$\begin{cases} u + w = 2rp \\ u - w = 2tp \end{cases}$ using this system of equations, we see:

$2u = 2rp + 2tp \rightarrow u = p(r + t)$ and

$2w = 2rp - 2tp \rightarrow w = p(r - t)$.

This shows $p|u$ and $p|w$ which is a contradiction to $\gcd(u, w) = 1$. Thus, gcd $\left(\frac{u+w}{2}, \frac{u-w}{2}\right) = 1$.

Proposition 2: Let $a, b, c, \in \mathbb{Z}^+$ such that $a^2 = bc$ and $\gcd(b, c) = 1$. Then b and c are perfect squares.

Proof. Let *a* be expressed as the product of primes $p_n^{r_n}$ where p_i are unique primes and $r_i \in \mathbb{Z}^+$.

Let $a = p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n} \rightarrow a^2 = \left(p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}\right)^2 = p_1^{2r_1} \cdot p_2^{2r_2} \cdots p_n^{2r_n} \rightarrow a^2 = bc$

By substitution $bc = p_1^{2r_1} \cdot p_2^{2r_2} \cdots p_n^{2r_n}$ and since b and c are relatively prime, without loss of generality we can express b and c as the product of primes $b = p_1^{2r_1} \cdot p_2^{2r_2} \cdots p_k^{2r_k}$ and $c = p_{k+1}^{2r_{k+1}} \cdot p_{k+2}^{2r_{k+2}} \cdots p_n^{2r_n}$.

Thus $b = \left(p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}\right)^2$ and $c = \left(p_{k+1}^{r_{k+1}} \cdot p_{k+2}^{r_{k+2}} \cdots p_n^{r_n}\right)^2$ where

$p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k} = m$ and $p_{k+1}^{r_{k+1}} \cdot p_{k+2}^{r_{k+2}} \cdots p_n^{r_n} = n$ for $m, n \in \mathbb{Z}^+$.

Thus $b = m^2, c = n^2$ and $a^2 = bc = m^2 n^2$.

Case 1: u is odd, v is even, and w is odd

Let us presume v is even, then $v = 2k, k \in \mathbb{Z}$ and we have

$$3v^2 = 12k^2 = w^2 - u^2 = (w+u)(w-u).$$

We also have: $\gcd(u, v) = \gcd(u, w) = \gcd(v, w) = 1$.

Since u and w are both odd, then $(w+u)$ and $(w-u)$ are both even.

Thus $3k^2 = \left(\frac{w+u}{2}\right)\left(\frac{w-u}{2}\right)$, where $\frac{w+u}{2}$ and $\frac{w-u}{2}$ are integers.

Since we already showed in Proposition 1 that $\left(\frac{w+u}{2}\right)$ & $\left(\frac{w-u}{2}\right)$ are relatively prime, we can say 3 divides either $\frac{w+u}{2}$ or $\frac{w-u}{2}$, but not both.

Let us presume $3 | \frac{w+u}{2}$, then the implication is that $k^2 = \frac{w-u}{2} \cdot \frac{w+u}{6}$ where $\frac{w-u}{2}, \frac{w+u}{6} \in \mathbb{Z}$ and $\gcd\left(\frac{w-u}{2}, \frac{w+u}{6}\right) = 1$. By Proposition 2, we know $\frac{w-u}{2}$ and $\frac{w+u}{6}$ are prefect squares.

Let $\frac{w-u}{2} = n^2$ and $\frac{w+u}{6} = m^2$, then $w - u = 2n^2, w + u = 6m^2$ by systems of equations we see:

$$\begin{cases} w - u = 2n^2 \\ w + u = 6m^2 \end{cases} \rightarrow 2w = 2n^2 + 6m^2 \rightarrow w = 3m^2 + n^2$$

Similarly, $\begin{cases} w - u = 2n^2 \\ w + u = 6m^2 \end{cases} \rightarrow 2u = 6m^2 - 2n^2 \rightarrow u = 3m^2 - n^2$

By substitution where $3v^2 = (w-u)(w+u) \rightarrow 3v^2 = (6m^2)(2n^2) = 12m^2n^2 \rightarrow$

$v^2 = 4m^2n^2 \rightarrow v = 2mn.$

What if $3|\frac{w-u}{2}$? Then the implication is that $k^2 = \frac{w-u}{6} \cdot \frac{w+u}{2}$ where $\gcd\left(\frac{w-u}{6}, \frac{w+u}{2}\right) = 1$ and $\frac{w-u}{6}, \frac{w+u}{2} \in \mathbb{Z}$. Again, by Proposition 2 we know $\frac{w-u}{6}$ and $\frac{w+u}{2}$ are prefect squares.

Let $\frac{w-u}{6} = m^2$ and $\frac{w+u}{2} = n^2$ then $w - u = 6m^2, w + u = 2n^2$ by systems of equations we see:

$\begin{cases} w - u = 6m^2 \\ w + u = 2n^2 \end{cases} \rightarrow 2w = 6m^2 + 2n^2 \rightarrow w = 3m^2 + n^2$

Similarly, $\begin{cases} w - u = 6m^2 \\ w + u = 2n^2 \end{cases} \rightarrow 2u = 2n^2 - 6m^2 \rightarrow u = n^2 - 3m^2$

By substitution where $3v^2 = (w-u)(w+u) \rightarrow 3v^2 = (6m^2)(2n^2) = 12m^2n^2 \rightarrow$

$v^2 = 4m^2n^2 \rightarrow v = 2mn$

Here we conclude, $u = \pm(3m^2 - n^2), v = \pm 2mn, w = \pm(3m^2 + n^2)$ seeing that m and n are relatively prime non-negative integers of opposite parity. Note that since we are only interested in u and v, we can write $u = (3m^2 - n^2), v = 2mn$, where m and n are relatively prime integers of opposite parity.

Case 2: u is odd, v is odd, and w is even

As in case 1 we have: $\gcd(u, v) = \gcd(u, w) = \gcd(v, w) = 1$ and we write

$3v^2 = w^2 - u^2 = (w+u)(w-u).$

Let us presume $3|w + u$ then, the implication is that $v^2 = \left(\frac{w+u}{3}\right)(w - u)$ where $\left(\frac{w+u}{3}\right)$ and $(w - u)$ are relatively prime integers. By Proposition 2 we know $\left(\frac{w+u}{3}\right)$ and $(w - u)$ are prefect squares. Let $\left(\frac{w+u}{3}\right) = m^2, (w - u) = n^2$ where $m, n \in \mathbb{Z}$ then $u = \frac{3m^2 - n^2}{2}$,

$v = mn$, and $w = \frac{3m^2 + n^2}{2}$.

What if $3|w - u$? Then the implication is $v^2 = \left(\frac{w-u}{3}\right)(w + u)$ where $\left(\frac{w-u}{3}\right)$ and

$(w+u)$ are relatively prime integers. Again, by Proposition 2 we know $\left(\frac{w-u}{3}\right)$ and $(w+u)$ are prefect squares. Let $\left(\frac{w-u}{3}\right) = m^2, (w+u) = n^2$ where $m, n \in \mathbb{Z}$ then $u = \frac{3m^2-n^2}{2}$, $v = mn$, and $w = \frac{3m^2+n^2}{2}$.

Description of $f(x)$

Finally, let us obtain a description for $f(x)$ in terms of u and v. Recall, $g(x) = f\left(x + \frac{a+b}{2}\right) = (x^2 - t^2)(x - s)$ where $t = \frac{b-a}{2}$ and $s = d - \frac{a+b}{2}$ and the roots of $g(x)$ are $\pm t, s$. In other words, we can express a and b in terms of r, s, and t.

$$\begin{cases} s = d - \dfrac{a+b}{2} \\ t = \dfrac{b-a}{2} \end{cases} \rightarrow \begin{cases} s + t = d - a \\ s - t = d - b \end{cases} \rightarrow \begin{cases} a = d - s - t \\ b = d - s + t \end{cases}$$

By substitution $f(x) = (x-a)(x-b)(x-d) = (x - d + s + t)(x - d + s - t)(x - d)$.

In terms of d, s, t for $f(x)$ we get:

$$f(x) = x^3 - (3d - 2s)x^2 + (3d^2 - 4ds + s^2 - t^2)x - (d^3 - 2d^2s + ds^2 - dt^2)$$

Now let us substitute $s = \alpha u, t = \alpha v$ into $f(x)$ where $\alpha \in \mathbb{Q}$.

$$f(x) = x^3 - (3d - 2\alpha u)x^2 + (3d^2 - 4\alpha du + \alpha^2u^2 - \alpha^2v^2)x - (d^3 - 2\alpha d^2u + \alpha^2du^2 - \alpha^2d\,v^2)$$

Together with the description of all primitive solutions (u, v, w) of the equation $u^2 + 3v^2 = w^2$ this gives us a description of all rational cubic polynomials $f(x)$ such that the roots of its first and second derivative are also rational.

Examples

Given $f(x)$ in terms of $\alpha, d, u, v \in \mathbb{Q}$. Let $\alpha = 1, d = 1, u, v \in \mathbb{Z}$ as determined by m and n where $m$and n are relatively prime have opposite parity or both m and n are odd.

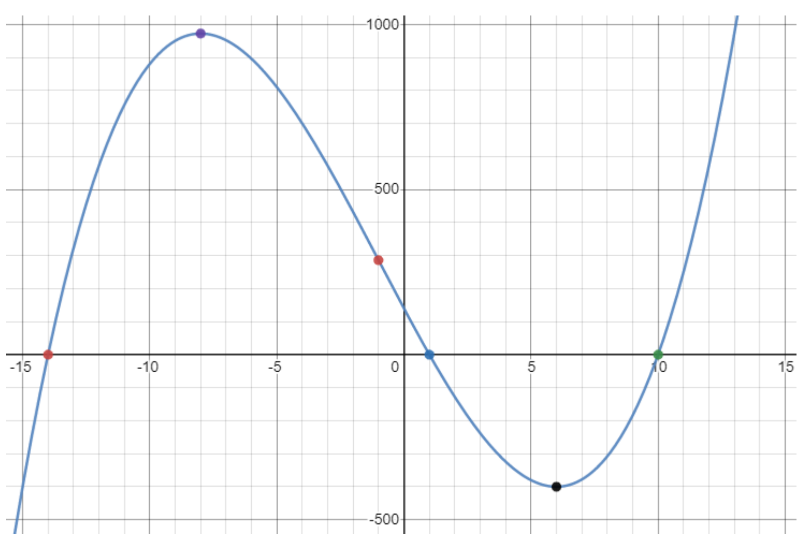
$$f(x) = x^3 - (3d - 2\alpha u)x^2 + (3d^2 - 4\alpha du + \alpha^2u^2 - \alpha^2v^2)x - (d^3 - 2\alpha d^2u + \alpha^2du^2 - \alpha^2d\,v^2)$$

$$f(x) = x^3 - (3 - 2u)x^2 + (3 - 4u + u^2 - v^2)x + (2u - u^2 + v^2)$$

m	n	$u = 3m^2 - n^2$	$v = 2mn$	$f(x) = x^3 + bx^2 + cx + d$
2	1	11	4	$f(x) = x^3 + 19x^2 + 64x - 84$
2	3	3	12	$f(x) = x^3 + 3x^2 - 144x + 140$
1	2	-1	2	$f(x) = x^3 - 5x^2 + 4x$
3	4	11	24	$f(x) = x^3 + 19x^2 - 496x + 476$

m	n	$u = \frac{3m^2 - n^2}{2}$	$v = mn$	$f(x) = x^3 + bx^2 + cx + d$
1	3	-3	3	$f(x) = x^3 - 9x^2 + 15x - 6$
3	7	-11	21	$f(x) = x^3 + 25x^2 - 273x - 297$
5	3	33	15	$f(x) = x^3 - 63x^2 + 735x + 799$
3	5	1	15	$f(x) = x^3 - x^2 - 256x - 256$

Let us look at an example in more depth. For $m = 2, n = 3$ we obtain $u = 3, v = 12$ thus generating $f(x) = x^3 + 3x^2 - 144x + 140$ The factored form of f is $f(x) = (x + 14)(x - 1)(x - 10)$ where the roots are $x = -14, 1, 10$. Then $f'(x) = 3x^2 + 6x - 144 = 3(x^2 + 2x - 48) = 3(x + 8)(x - 6)$ where the roots of $f'(x)$ are $x = -8, 6$ which indicate the local extrema of f. Additionally, we get $f"(x) = 6x + 6$ where the root is $x = -1$ which indicates the point of inflection of f. Finally, the x-intercepts of $f(x)$ are $(-14, 0), (1, 0)$, and $(10, 0)$. The local maximum is $(-8, 972)$, local minimum is $(6, -400)$, and the point of inflection is $(-1,286)$.



Conclusion

Cubic polynomials with rational roots and critical values are extremely valuable to mathematics educators at multiple levels of education. These polynomials can be used to present various topics to students without having the students getting tangled in heavy algebra computations. For example, students can more easily graph, find key values such as roots, local extrema, and inflection points without the need for a calculator. Their work with these problems can expand on basic concepts without heavy calculator usage which can be helpful at times when we as educations want students to internalize their knowledge and work.

References

Gupta, S.& Szymanski, W. (2010). Cubic Polynomials with Rational Roots and Critical Points. *The College Mathematics Journal, 41(5)*, 365-369. https://doi.org/10.4169/074683410X521964